Solving the Model

The base case model features a representative household that chooses paths of consumption, leisure, and investment to maximize utility. The paths of TFP and population are exogenously given (in database.txt), and the agent has perfect foresight over their values. We start the model at date T_0 = 1980 and let time run out to infinity.

Definition. Given sequences of productivity, A_t, and working age population, N_t, for $t=T_0$, $T_0+1,...$, and the initial capital stock, K_{T_0}, an *equilibrium* is sequences of wages, w_t, interest rates, r_t, consumption, C_t, labor, L_t, and capital stocks, K_t, such that

1. given the wages and interest rates, the representative household chooses consumption, labor, and capital to maximize the utility function (1) subject to the budget constraints (2), appropriate nonnegativity constraints (3), and the constraint on K_{T_0} (4);

(1) $$\max_{c,l,k} \sum_{t=T_0}^{\infty} \beta^t \left(\gamma \log C_t + (1-\gamma)\log(\bar{h}N_t - L_t)\right)$$

s.t. $\forall t$

(2) $$C_t + K_{t+1} \leq w_t L_t + (1-\delta + r_t)K_t$$

(3) $$C_t, K_t, L_t \geq 0, \quad L_t \leq \bar{h}N_t$$

(4) $$K_{T_0} \text{ given}$$

2. the wages and interest rates, together with the firms' choices of labor and capital, satisfy the cost minimization and zero profit conditions;

(5) $$w_t = (1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha}$$

(6) $$r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha}$$

3. consumption, labor, and capital satisfy the feasibility condition.

(7) $$C_t + K_{t+1} = A_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta)K_t$$

We turn these equilibrium conditions into a system of equations that can be solved to find the equilibrium of the model. We begin by taking the first-order conditions of the household's problem to obtain

(8) $$w_t(\bar{h}N_t - L_t) = \frac{1-\gamma}{\gamma}C_t$$

(9) $$\frac{C_{t+1}}{C_t} = \beta(1-\delta + r_{t+1})$$

Combining the household's optimality conditions (8) and (9), the firm optimality conditions (5) and (6), and the feasibility condition (7), we can specify a system of equations that can be solved to find the equilibrium of the model.

Plugging (5) and (6) into (8) and (9), and using the feasibility condition (7), we obtain the system of equations

(10) $$(1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha}(\bar{h}N_t - L_t) = \frac{1-\gamma}{\gamma}C_t$$

(11) $$\frac{C_{t+1}}{C_t} = \beta(1-\delta + \alpha A_{t+1}K_{t+1}^{\alpha-1}L_{t+1}^{1-\alpha})$$

(12) $$C_t + K_{t+1} = A_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta)K_t$$

Solving for an equilibrium involves choosing sequences of consumption, capital stocks, and hours worked such that these equations are satisfied, given the initial condition K_{T_0} and final condition, the transversality condition,

(13) $$\lim_{t\to\infty} \beta^t \frac{\gamma}{C_t} K_{t+1} = 0$$

In principle, the system of equations that characterize the equilibrium, (10)-(12), involves an infinite number of equations and unknowns. To make the computation of an equilibrium tractable, we assume that the economy converges to the balanced-growth path at some date T_1, which allows us to truncate the system of equations. Using the feasibility condition (12) to solve for C_t, we can write these equations as

(14) $$(1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha}(\bar{h}N_t - L_t) = \frac{1-\gamma}{\gamma}(A_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta)K_t - K_{t+1}),\ t = T_0, T_0+1, ..., T_1$$

(15) $$\frac{A_{t+1}K_{t+1}^{\alpha}L_{t+1}^{1-\alpha} + (1-\delta)K_{t+1} - K_{t+2}}{A_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta)K_t - K_{t+1}} = \beta(1-\delta + \alpha A_{t+1}K_{t+1}^{\alpha-1}L_{t+1}^{1-\alpha}),\ t = T_0, T_0+1, ..., T_1 - 1$$

where $K_{T_1+1} = g\eta K_{T_1}$.

We choose T_1 so that $T_1 - T_0$ is large, say 60, so that we are solving the model over the period 1980–2040. We then construct the exogenous variables. The exogenous variables A_t, N_t for 1980–2005 are as they are in the data. For 2006–2040, we assume that TFP grows at a constant rate equal to the average growth rate of TFP over the period 1980–

2005 and that the working-age population grows at the same rate as in 2004–2005. These are the growth rates $g^{1-\alpha}$ and η in the specification of the balanced-growth path.

Solving the model now consists of choosing $K_{T_0+1}, K_{T_0+2}, ..., K_{T_1}$, and $L_{T_0}, L_{T_0+1}, ..., L_{T_1}$ to solve the system of equations (14) and (15). This system of $2(T_1 - T_0 + 1) - 1$ nonlinear equations in $2(T_1 - T_0 + 1) - 1$ unknowns can be solved relatively quickly using numerical methods.

Running the Program

Program Inputs
The user must provide two files to the program. The first file should be named "paramBase.txt" and consist of a single column vector of the parameters β, γ, δ, α, *g*, η, and K_{T_0}. The second file should be named "dataBase.txt" and contain a $(T_1 - T_0) \times 6$ matrix of values: levels of TFP, A_t, working-age population, N_t, available hours, $\bar{h}N_t$, consumption tax rates, labor tax rates, and capital tax rates. These files must be in a form that can be interpreted by MATLAB. One method is to enter the data into a Microsoft Excel spreadsheet and save the file as a tab delimited file.

Program Output
Upon successful completion, the program will save a $(T_1 - T_0) \times 6$ matrix of values to the file "output.xls" which is a tab delimited file. This file can be opened in Excel for inspection or to create plots. The data can also be directly manipulated in MATLAB. The variables in the file are $\mathrm{Y_t/N_t}$, $\mathrm{X_t/Y_t}$, $\mathrm{L_t/(\bar{h}N_t)}$, $\mathrm{C_t/Y_t}$, $\mathrm{K_t/Y_t}$ and $\mathrm{r_t}$-δ .

Solution Method
Choosing $K_{T_0+1}, K_{T_0+2}, ..., K_{T_1}$ and $L_{T_0}, L_{T_0+1}, ..., L_{T_1}$ to satisfy (14) for $t = T_0, T_0 + 1, ..., T_1$, and (15) for $t = T_0, T_0 + 1, ..., T_1 - 1$ requires solving $2(T_1 - T_0 + 1) - 1$ equations in $2(T_1 - T_0 + 1) - 1$ unknowns. The accompanying MATLAB program uses Newton's method to solve the system of equations.

Define the stacked vector of variables $x = [K_{T_0+1}, K_{T_0+2}, ..., K_{T_1}, L_{T_0}, L_{T_0+1}, ..., L_{T_1}]$ $x = [$ and arrange the system of equations so that they are of the form $f(x) = \vec{0}$, where $\vec{0}$ is a $2(T_1 - T_0 + 1) - 1$ vector of zeros. The algorithm involves making an initial guess at the variables, x^0, and updating the guess by $x^{i+1} = x^i - Df(x^i)^{-1} f(x^i)$, where $Df(x^i)$ is the matrix of partial derivatives of $f(x)$ evaluated at x^i. The system of equations does not have closed-form expressions for the partial derivatives needed to compute $Df(x^i)$, and so the derivatives have to be evaluated numerically. A solution is obtained when the function, evaluated at the new iterate of x, has a maximum error less than some value ε, where ε is a small number.

Although this method of solving a system of nonlinear equations can converge to a solution quickly, this method is not globally convergent and can become stuck away form a zero of $f(x)$ or may not converge at all. The initial guess, x^0, is important. Further details on the implementation of Newton's method can be found in Press, Flannery, Teukolsky, and Vetterling (2002).

To increase the probability of the algorithm converging to the correct answer, we solve a sequence of models, beginning with a simple version of the model, which we know how to solve, and progressing to the model that we would like to solve. The first model we solve is the one in which TFP, population, and available hours are constant and equal to their average values from 1980 to 2005, and the tax rates are all zero. The solution to this problem is relatively easy to find. The next model takes TFP, population, available hours, and tax rates to be convex combinations of the constant values used in the initial model and the actual values of TFP, population, available hours, and tax rates from the data. Let λ be the weight on the constant values, so that $(1-\lambda)$ is the weight on the values from the data. The algorithm requires repeatedly decrementing λ and solving the resulting model, each time using the solution to the model before it as the initial guess. The algorithm proceeds until it solves the case in which $\lambda = 0$, which corresponds to the model whose solution we desire.